EPFL– Fall 2025
Series 8

Differential Geometry IV: General relativity

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8.1 (a) Let $\mathcal{D}$ be the domain in the Einstein cylinder $(\mathbb{R}\times\mathbb{S}^3, g_E)$, $g_E=-dt^2+g_{\mathbb{S}^3}$, which is the image of Minkowski spacetime $(\mathbb{R}^{3+1},\eta)$ under the conformal map $(u,v)\to(\mathrm{Arctan}(u),\mathrm{Arctan}(v))$ that we saw in class. Let p be a point on future null infinity $\mathcal{I}^+\subset\partial\mathcal{D}$ and consider the set $C^-(p)\cap\mathcal{D}$ of past null geodesics emanating from p restricted to $\mathcal{D}$. How does this set look like in the standard Cartesian coordinates of $\mathbb{R}^{3+1}$? Deduce that every pair of null geodesics of $(\mathbb{R}^{3+1},\eta)$ asymptoting in the future to the same point on $\mathcal{I}^+$ have to asymptote in the past to the same point on $\mathcal{I}^-$.

(b) Let $(\mathcal{M}^{3+1},g)$ be a spherically symmetric spacetime without axis, i.e. $\mathcal{M}^{3+1}=\mathcal{U}^{1+1}\times\mathbb{S}^2$ and, in any local coordinates (x^1,x^2) on $\mathcal{U}$,

$$g=\tilde{g}_{AB}dx^Adx^B+r^2g_{\mathbb{S}^2},$$

where $\tilde{g}$ is a Lorentzian metric on $\mathcal{U}$ and $r:\mathcal{U}\to(0,+\infty)$ is a smooth function. The spacetime $(\mathcal{U},\tilde{g})$ is known as the *Penrose diagram* of $(\mathcal{M},g)$ and can be formally thought of as the projection $\mathcal{U}=\mathcal{M}/SO(3)$. Show that the image $\tilde{\gamma}$ in $\mathcal{U}$ of a causal curve γ in $\mathcal{M}$ is again a causal curve (hence, Penrose diagrams are useful 2-dimensional tools to read-off the causal structure of a 4 dimensional spacetime). In which case is the projection of a null curve in $\mathcal{M}$ again a null curve in $\mathcal{U}$?

Solution. (a) One can easily verify that $C^-(p)\cap\mathcal{D}$ is simply a null hyperplane of the form $\mathcal{H}_\omega=\{(t,x^1,x^2,x^3):\quad t=\omega_ix^i\}$, where $\|\omega\|=1$ (i.e. $(1,\omega)\in\mathbb{R}^{1+3}$ is a null vector, the generator of $\mathcal{H}_\omega$). In particular, $C^-(p)\cap\mathcal{D}$ consists of all parallel null straight lines lying in the same hyperplane. By time reversal symmetry, the same is true for $C^+(q)\cap\mathcal{D}$ for any $q\in\mathcal{I}^+$.

(b) Let $\gamma:I\to\mathcal{M}$ be a causal curve, namely $\dot{\gamma}\neq0$ and

$$g(\dot{\gamma},\dot{\gamma})\leqslant0.$$

If we express $\gamma(t)=(\tilde{\gamma}(t),\bar{\gamma}(t))$ in the decomposition $\mathcal{M}=\mathcal{U}\times\mathbb{S}^2$, then the product form of the metric means that the above relation is equivalent to

$$\tilde{g}(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}})+r^2g_{\mathbb{S}^2}(\dot{\bar{\gamma}},\dot{\bar{\gamma}})\leqslant0.$$

Since $g_{\mathbb{S}^2}$ is Riemannian, we get that

$$\tilde{g}(\dot{\tilde{\gamma}},\dot{\tilde{\gamma}})\leqslant-r^2g_{\mathbb{S}^2}(\dot{\bar{\gamma}},\dot{\bar{\gamma}})\leqslant0,$$

i.e. $\tilde{\gamma}$ is a causal curve of $\mathcal{U}$. If γ is null, then $\tilde{\gamma}$ is null only when $\dot{\bar{\gamma}}$ vanishes, i.e. when γ is radial (it has constant projection on $\mathbb{S}^2$).

8.2 (a) Show that the Schwarzschild metric

$$g_M=-\Big(1-\frac{2M}{r}\Big)dt^2+\Big(1-\frac{2M}{r}\Big)^{-1}dr^2+r^2\big(d\theta^2+\sin^2\theta d\phi^2\big)\tag{1}$$

on $\mathcal{M}=\mathbb{R}_t\times(2M,+\infty)_r\times\mathbb{S}^2$ is indeed a solution of the vacuum Einstein equations.

(b) Show that $(\mathcal{M}, g_M)$ embeds isometrically into $\tilde{\mathcal{M}} = \mathbb{R}_{t^*} \times (0, +\infty)_r \times \mathbb{S}^2$ with

$$g_{\tilde{\mathcal{M}}} = -\Big(1 - \frac{2M}{r}\Big)(dt^*)^2 + \frac{4M}{r} dt^* dr + \Big(1 + \frac{2M}{r}\Big) dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)$$

(*Hint: Use the coordinate transformation* $t^* = t + f(r)$ *for a suitable* $f(r)$*.*) Show that, in the extended spacetime, the region $\{r \leqslant 2M\}$ corresponds to a *black hole*, that is to say, no future directed causal curve starting from $\{r \leqslant 2M\}$ can end up in the asymptotically region $\{r \gg 1\}$.

Solution. (a) In the (t, r, θ, ϕ) coordinates on $\mathcal{M}$, the matrix $[(g_M)_{\mu\nu}]$ of the Schwarzschild metric takes the diagonal form

$$g_M = \begin{pmatrix} -\Big(1 - \frac{2M}{r}\Big) & 0 & 0 & 0 \\ 0 & \Big(1 - \frac{2M}{r}\Big)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

and, therefore, the inverse matrix $[(g_M)_{\mu\nu}]^{-1} = [g_M^{\mu\nu}]$ is

$$g_M^{-1} = \begin{pmatrix} -\Big(1 - \frac{2M}{r}\Big)^{-1} & 0 & 0 & 0 \\ 0 & \Big(1 - \frac{2M}{r}\Big) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2}\theta \end{pmatrix}.$$

We can thus compute the Christoffel symbols of g through the formula

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}\big(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\big)$$

as follows (note that, since $(g_M)_{\mu\nu}$ is independent of t, ϕ, any Christoffel symbol which only contains ∂_t and ∂_ϕ derivatives of g_M, such as $\Gamma^t_{t\phi}$, vanishes; note also that the only metric component depending non-trivially on θ is $g_{\phi\phi}$):

$$\Gamma^r_{tt} = \frac{M}{r^2}\Big(1 - \frac{2M}{r}\Big), \quad \Gamma^r_{rr} = -\frac{M}{r^2}\Big(1 - \frac{2M}{r}\Big)^{-1}, \quad \Gamma^r_{\theta\theta} = -(r - 2M), \quad \Gamma^r_{\phi\phi} = -(r - 2M)\sin^2\theta,$$
$$\Gamma^t_{tr} = \frac{M}{r^2}\Big(1 - \frac{2M}{r}\Big)^{-1},$$
$$\Gamma^\theta_{r\theta} = r^{-1}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,$$
$$\Gamma^\phi_{r\phi} = r^{-1}, \quad \Gamma^\phi_{\theta\phi} = \cot\theta,$$

and the rest of the Christoffel symbols vanish. For convenience, let us also compute the following expressions (recall that repeated indices are assumed to be summed over):

$$\Gamma^\alpha_{\alpha t} = 0, \quad \Gamma^\alpha_{\alpha r} = \frac{2}{r}, \quad \Gamma^\alpha_{\alpha\theta} = \cot\theta, \quad \Gamma^\alpha_{\alpha\phi} = 0.$$

Using the expression

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\kappa}\Gamma^{\kappa}_{\beta\delta} - \Gamma^{\alpha}_{\delta\kappa}\Gamma^{\kappa}_{\beta\gamma}$$

for the components of the Riemann curvature tensor, we readily obtain the following general expression for the components of the Ricci tensor $Ric_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$:

$$Ric_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{\mu\nu} - \Gamma^{\alpha}_{\nu\kappa}\Gamma^{\kappa}_{\alpha\mu}.$$

Thus, we can readily calculate in the case of the Schwarzschild metric using our calculation of the Christoffel symbols above (and ignoring the vanishing Christoffel symbols in the corresponding sums involving repeated indices) all the components of the Ricci tensor:

$$\begin{aligned}
Ric_{tt} &= \partial_\alpha \Gamma^{\alpha}_{tt} - \partial_t \Gamma^{\alpha}_{\alpha t} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{tt} - \Gamma^{\alpha}_{t\kappa}\Gamma^{\kappa}_{\alpha t} \\
&= \partial_r \Gamma^{r}_{tt} + \Gamma^{\alpha}_{\alpha r}\Gamma^{r}_{tt} - \Gamma^{r}_{tt}\Gamma^{t}_{rt} - \Gamma^{t}_{tr}\Gamma^{r}_{tt} \\
&= \partial_r\Big(\frac{M}{r^2}\big(1-\frac{2M}{r}\big)\Big) + \frac{2}{r}\cdot\frac{M}{r^2}\big(1-\frac{2M}{r}\big) - 2\frac{M}{r^2}\big(1-\frac{2M}{r}\big)\cdot\frac{M}{r^2}\big(1-\frac{2M}{r}\big)^{-1} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
Ric_{tr} &= \partial_\alpha \Gamma^{\alpha}_{tr} - \partial_r \Gamma^{\alpha}_{\alpha t} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{tr} - \Gamma^{\alpha}_{r\kappa}\Gamma^{\kappa}_{\alpha t} \\
&= 0
\end{aligned}$$

$$Ric_{t\theta} = Ric_{t\phi} = 0$$

$$\begin{aligned}
Ric_{rr} &= \partial_\alpha \Gamma^{\alpha}_{rr} - \partial_r \Gamma^{\alpha}_{\alpha r} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{rr} - \Gamma^{\alpha}_{r\kappa}\Gamma^{\kappa}_{\alpha r} \\
&= \partial_r \Gamma^{r}_{rr} - \partial_r \Gamma^{r}_{rr} - \partial_r \Gamma^{t}_{tr} - \partial_r \Gamma^{\theta}_{\theta r} - \partial_r \Gamma^{\phi}_{\phi r} + \Gamma^{\alpha}_{\alpha r}\Gamma^{r}_{rr} - (\Gamma^{r}_{rr})^2 - (\Gamma^{t}_{tr})^2 - (\Gamma^{\theta}_{\theta r})^2 - (\Gamma^{\phi}_{\phi r})^2 \\
&= 0 - \partial_r\Big(-\frac{M}{r^2}\big(1-\frac{2M}{r}\big)^{-1}\Big) - 2\partial_r\Big(\frac{1}{r}\Big) \\
&\qquad + \frac{2}{r}\cdot\Big(-\frac{M}{r^2}\big(1-\frac{2M}{r}\big)^{-1}\Big) - \Big(-\frac{M}{r^2}\big(1-\frac{2M}{r}\big)^{-1}\Big)^2 - \Big(\frac{M}{r^2}\big(1-\frac{2M}{r}\big)^{-1}\Big)^2 - 2\Big(\frac{1}{r}\Big)^2 \\
&= 0
\end{aligned}$$

$$Ric_{r\theta} = Ric_{r\phi} = 0$$

$$\begin{aligned}
Ric_{\theta\theta} &= \partial_\alpha \Gamma^{\alpha}_{\theta\theta} - \partial_\theta \Gamma^{\alpha}_{\alpha\theta} + \Gamma^{\alpha}_{\alpha\kappa}\Gamma^{\kappa}_{\theta\theta} - \Gamma^{\alpha}_{\theta\kappa}\Gamma^{\kappa}_{\alpha\theta} \\
&= \partial_r \Gamma^{r}_{\theta\theta} - \partial_\theta \Gamma^{\phi}_{\phi\theta} + \Gamma^{\alpha}_{\alpha r}\Gamma^{r}_{\theta\theta} - 2\Gamma^{\theta}_{\theta r}\Gamma^{r}_{\theta\theta} - (\Gamma^{\phi}_{\phi\theta})^2 \\
&= \partial_r\big(-(r-2M)\big) - \partial_\theta \cot\theta + \frac{2}{r}\cdot\big(-(r-2M)\big) - \frac{2}{r}\big(-(r-2M)\big) - (\cot\theta)^2 \\
&= 0
\end{aligned}$$

$$Ric_{\theta\phi} = Ric_{\phi\phi} = 0.$$

(note that we can deduce that $Ric_{tA} = Ric_{rA} = 0$ for $A = \theta, \phi$ without doing the tedious computations, using simply the spherical symmetry of the metric: The quantities Ric_{tA}, Ric_{rA} for fixed t, r are simply vector fields on the sphere $\{t, r = const\}$ which have to be invariant under spherical rotations (since g_M has this property); however, any tangent vector at any point $p \in \mathbb{S}^2$ which is invariant under spherical rotations fixing p (and hence rotations of $T_p\mathbb{S}^2$) has to be the 0 vector. Similarly, Ric_{AB} for $A, B \in \{\theta, \phi\}$ has to be a symmetric $(0,2)$-tensor on the sphere which is invariant under rotations, thus $Ric_{\theta\theta} = 0$ implies that $Ric_{AB} = 0$ for $A, B \in \{\theta, \phi\}$).

(b) Using a coordinate transformation $(t, r, \theta, \phi) \to (t^*, r, \theta, \phi)$ for $t^*(t, r) = t + f(r)$ (for a smooth function $f : (2M, +\infty) \to \mathbb{R}$ to be determined shortly), we can readily calculate that, in the new coordinate system:

$$dt = dt^* - f'(r)dr, \quad , dr = dr, \quad d\theta = d\theta, \quad d\phi = d\phi.$$

Therefore, substituting in the expression for the metric, we obtain that in the (t^*, r, θ, ϕ) system g_M looks as follows:

$$\begin{aligned} g_M &= -\Big(1 - \frac{2M}{r}\Big)(dt^* - f'dr)^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2\big(d\theta^2 \sin^2\theta d\phi^2\big) \\ &= -\Big(1 - \frac{2M}{r}\Big)(dt^*)^2 + 2\Big(1 - \frac{2M}{r}\Big)f'(r)dt^*dr + \Big[-\Big(1 - \frac{2M}{r}\Big)(f'(r))^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}\Big]dr^2 + r^2\big(d\theta^2 \sin^2\theta \end{aligned}$$

Therefore, choosing $f(r) = 2M\log(r - 2M)$ in order to achieve

$$-\Big(1 - \frac{2M}{r}\Big)(f'(r))^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} = 1 + \frac{2M}{r},$$

we infer that

$$g_M = -\Big(1 - \frac{2M}{r}\Big)(dt^*)^2 + \frac{4M}{r}dt^*dr + \Big(1 + \frac{2M}{r}\Big)dr^2 + r^2\big(d\theta^2 \sin^2\theta d\phi^2\big).$$

Note that the original manifold $\mathcal{M}$ is identified in the (t^*, r, θ, ϕ) coordinates with $\mathbb{R}_{t^*} \times (2M, +\infty) \times \mathbb{S}^2$. Therefore, the inclusion map $\iota : \mathcal{M} = \mathbb{R}_{t^*} \times (2M, +\infty) \times \mathbb{S}^2 \to \mathbb{R}_{t^*} \times (0, +\infty) \times \mathbb{S}^2 = \tilde{\mathcal{M}}$, $\iota(t^*, r, \theta, \phi) = (t^*, r, \theta, \phi)$, is an isometry (since $(g_M)_{\mu\nu} = (g_{\tilde{\mathcal{M}}})_{\mu\nu}$ in these coordinates). It is also easy to verify that $\tilde{\mathcal{M}}$ covers precisely the regions $I + II$ of the maximally extended Schwarzschild spacetime: This can be seen, for instance, by noting that the coordinate transformation $(t^*, r) \to (v, r^*)$ where $r^* = r + 2M\log(r - 2M)$ and $v = t + r^*$ on $\mathcal{M}$, extends smoothly on the whole of $\tilde{\mathcal{M}}$; note that, as we saw in class, the maximal domain of the (v, r^*) coordinates covers precisely the regions $I + II$ of the maximally extended Schwarzschild spacetime. Alternatively, one can argue usiing the Penrose diagram of Schwarzschild and noting that $\tilde{\mathcal{M}}$ contains the maximal future extension of all radial future directed null geodesics emanating in $\mathcal{M}$, but that any radial past directed null geodesic that is inextendible in $\mathcal{M}$ is also inextendible in $\tilde{\mathcal{M}}$.

8.3 (a) Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a geodesic in the Schwarzschild (exterior) spacetime $(\mathcal{M}, g_M)$. Show that the geodesic equation takes the form

$$\begin{aligned} \frac{d}{ds}\Big(\big(1 - \frac{2M}{r}\big)\dot{t}\Big) &= 0, \\ \frac{d}{ds}\Big(\big(1 - \frac{2M}{r}\big)^{-1}\dot{r}\Big) &= \frac{1}{2}\Big(-\frac{2M}{r^2}\dot{t}^2 - \big(1 - \frac{2M}{r}\big)^{-2}\frac{2M}{r^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\Big), \\ \frac{d}{ds}\Big(r^2\dot{\theta}\Big) &= \frac{1}{2}r^2\sin\theta\cos\theta\dot{\phi}^2, \end{aligned}$$

$$\frac{d}{ds}\Big(r^2\sin^2\theta\dot{\phi}\Big)=0.$$

Deduce that one can without loss of generality one can consider geodesics lying in the equatorial plane $\theta=\frac{\pi}{2}$ (by possibly rotating the coordinate system (θ,ϕ) on $\mathbb{S}^2$.) Note that, in this case, the first and fourth of the equations above reduce to the statement that the energy E and angular momentum L of a geodesic are constant (i.e. are constants of motion for the geodesic flow).

(b) Show that there exist "trapped" null geodesics orbitting the black hole (i.e. null geodesics that never approach $r=2M$ or $r=\infty$) (*Hint: For an appropriately chosen value of* $r=r_0>2M$*, show that there exist null geodesics with* $r(s)=r_0$ *for all* s*..* Contrast this with the situation on Minkowski spacetime.
Remark. The region traced out by trapped null geodesics consists the so-called *photon sphere* of a black hole.

(c) Show that, for any $\mu>0$, there exist timelike geodesics γ in the Schwarzschild spacetime with $g(\dot{\gamma},\dot{\gamma})=-\mu$ which are trapped. (*Hint: It might be convenient, instead of working with the second order equations, to use the invariants of the geodesic flow and obtain a relation for* $\dot{r}$ *and observe that, for an appropriate choice of* E, L *and* $r(0)$*,* $r(s)$ *cannot escape a bounded interval in* r*.*)
Remark. These timelike orbits correspond to massive objects (e.g. planets) moving under the influence of gravity in Schwarzschild spacetime. Unlike the trapped null geodesics, these orbits are *stable*, namely they remain trapped even under small perturbations of the initial condition $r(0)$ and the conserved quantities E, L, μ (were these trapped orbits not stable, earth would plunge in the sun under small perturbations of its orbit).

Solution.

(a) We can readily express the geodesic equation $\ddot{x}^\mu+\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=0$ in the (t,r,θ,ϕ) coordinate system on Schwarzschild exterior using the calculation of the Christoffel symbols for the Schwarzschild metric in Exercise 7.1. Recall that we computed:

$$\Gamma^r_{tt}=\frac{M}{r^2}\Big(1-\frac{2M}{r}\Big),\quad \Gamma^r_{rr}=-\frac{M}{r^2}\Big(1-\frac{2M}{r}\Big)^{-1},\quad \Gamma^r_{\theta\theta}=-(r-2M),\quad \Gamma^r_{\phi\phi}=-(r-2M)\sin^2\theta,$$
$$\Gamma^t_{tr}=\frac{M}{r^2}\Big(1-\frac{2M}{r}\Big)^{-1},$$
$$\Gamma^\theta_{r\theta}=r^{-1},\quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta,$$
$$\Gamma^\phi_{r\phi}=r^{-1},\quad \Gamma^\phi_{\theta\phi}=\cot\theta,$$

and the rest of the Christoffel symbols vanish.

Let $s\to\gamma(s)=(t(s),r(s),\theta(s),\phi(s))$, be a geodesic with the property that $\theta(0)=\frac{\pi}{2}$ and $\dot{\theta}(0)=0$. We will show that $\theta(s)=\frac{\pi}{2}$ for all s. To this end, by the uniqueness property for solutions to the initial value problem for the geodesic equation, it suffices to show that the system of geodesic equation admits a solution of the form $\bar{\gamma}(s) = (t(s), r(s), \frac{\pi}{2}, \phi(s))$ with $\bar{\gamma}(0) = \gamma(0)$ and $\dot{\bar{\gamma}}(0) = \dot{\gamma}(0)$. Setting $\theta = \frac{\pi}{2}$ in the system of equations, we obtain the following reduced system for $(t(s), r(s), \phi(s))$:

$$\frac{d}{ds}\Big(\big(1-\frac{2M}{r}\big)\dot{t}\Big) = 0, \tag{2}$$

$$\frac{d}{ds}\Big(\big(1-\frac{2M}{r}\big)^{-1}\dot{r}\Big) = \frac{1}{2}\Big(-\frac{2M}{r^2}\dot{t}^2 - \big(1-\frac{2M}{r}\big)^{-2}\frac{2M}{r^2}\dot{r}^2 + 2r\dot{\phi}^2\Big), \tag{3}$$

$$\frac{d}{ds}\Big(r^2\dot{\phi}\Big) = 0. \tag{4}$$

Note that the equation for $\theta(s)$, namely

$$\frac{d}{ds}\Big(r^2\dot{\theta}\Big) = \frac{1}{2}r^2\sin\theta\cos\theta\dot{\phi}^2,$$

is trivialy satisified when $\theta(s) = \frac{\pi}{2}$. Thus, if $(t(s), r(s), \phi(s))$ is the (unique) solution of the ODE system (2) with the initial data at $s = 0$ associated to $(\gamma(0), \dot{\gamma}(0))$, then the curve $\bar{\gamma}(s) = (t(s), r(s), \theta(s) = \frac{\pi}{2}, \phi(s))$ is a solution of the original geodesic system of equations.

If $s \to \gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ is a geodesic with $(\theta(0), \phi(0)) = (\theta_0, \phi_0) = p \in \mathbb{S}^2$ and $(\dot{\theta}(0), \dot{\phi}(0)) = (\theta_1, \phi_1) = v \in T_{(\theta_0,\phi_0)}\mathbb{S}^2$, then by changing the coordinate system on $\mathbb{S}^2$ through an appropriate rotation (chosen in a way so that the point p lies on the equator $\theta' = \frac{\pi}{2}$ and the tangent vector v is parallel to the same equator) we end up in the simpler case $\theta_0 = \frac{\pi}{2}$, $\theta_1 = 0$. Therefore, when studying the properties of a single geodesic γ on Schwarzschild spacetimes, we can always assume that the coordinate system on the spheres of symmetry has been chosen so that γ lies in the equatorial plane $\theta = \frac{\pi}{2}$.

(b) Let $s \to \gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a geodesic. Without loss of generality, we can assume (as explained in the previous part of this exercise) that $\theta(s) = \frac{\pi}{2}$. The quantities

$$\mu = -g(\dot{\gamma}, \dot{\gamma}), \quad E = -g(\dot{\gamma}, \partial_t), \quad L = g(\dot{\gamma}, \partial_\phi),$$

are constant along γ (μ is constant along a geodesic on any Lorentzian manifold, while E and L are the conserved quantities associated to the Killing vector fields ∂_t and ∂_ϕ). Using the expression of the Schwarzschild metric in the (t, r, θ, ϕ) coordinates, we calculate

$$E = \Big(1-\frac{2M}{r}\Big)\dot{t}, \quad L = r^2\dot{\phi}$$

and, therefore,

$$\mu = \Big(1-\frac{2M}{r}\Big)\dot{t}^2 - \Big(1-\frac{2M}{r}\Big)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \frac{E^2}{1-\frac{2M}{r}} - \Big(1-\frac{2M}{r}\Big)^{-1}\dot{r}^2 - \frac{L^2}{r^2},$$

which can be reexpressed as

$$\dot{r}^2 = E^2 - \Big(1-\frac{2M}{r}\Big)\Big(\frac{L^2}{r^2} + \mu\Big). \tag{5}$$

The second order ODE for $r(s)$ takes the following form in terms of the conserved quantities E, L, μ:

$$\frac{d}{ds}\Big(\big(1-\frac{2M}{r}\big)^{-1}\dot{r}\Big) = -\frac{2M}{(r-2M)^2}E^2 + \frac{(r-M)}{r^3(r-2M)}L^2 + \frac{M}{r(r-2M)}\mu. \tag{6}$$

In the case when γ is a null geodesic, we have $\mu = 0$ and $E \neq 0$ (since the inner product of two non-zero causal vectors cannot vanish). Thus, the relation (5) takes the form

$$\dot{r}^2 = E^2\Big(1 - V(r)\frac{L^2}{E^2}\Big),$$

where $V(r) = \frac{1}{r^2} - \frac{2M}{r^3}$. Note that the function V has a local maximum $V(r_0) = \frac{1}{27M^2}$ at $r_0 = 3M$. Therefore, if we choose the initial data of the geodesic so that $E^2 = V(r_0)L^2$ and $r(0) = r_0$, then the above relation implies that $\dot{r}(0) = 0$. In this case, it is easy to verify that for those values of E, L, μ and r, the right hand side of the ODE (6) also vanishes. Therefore, the unique solution of the corresponding initial value problem is going to be $r(s) = r_0 = 3M$. Notice that this is a null geodesic that neither escapes to $r = +\infty$ nor crosses the event horizon at $r = 2M$.

It is also easy to verify that there is no value $r_1 \neq r_0$ such that there exists a null geodesic with $r(s) = const = r_1$ (this is because for such a value of r the right hand sides of both (5) and (6) have to vanish); in view of the fact that the hypersurface $r = r_0$ contains the orbits of "trapped photons", it is called the photon sphere of the Schwarzschild black hole.

(c) Let us examine the relation (5) for $\mu > 0$. Note that this relation takes the form

$$\dot{r}^2 = E^2 - F_{L,\mu}(r),$$

with

$$F_{L,\mu}(r) = \Big(1 - \frac{2M}{r}\Big)\Big(\frac{L^2}{r^2} + \mu\Big).$$

Our aim is to show that, in this case, there exist appropriate choices of E, L such that the function $F_{L,\mu}$ above has the property that there exist $r_2 > r_1 > 2M$ such that $E^2 - F_{L,\mu}(r) \geqslant 0$ for $r \in [r_1, r_2]$ and $E^2 - F_{L,\mu}(r) < 0$ for $r \in [r_1 - \delta) \cup (r_2, +\delta)$. If this is indeed the case, then it would immediately follow that any geodesic $\gamma(s)$ with these values for its conserved quantities starting from a point with $r(0) \in [r_1, r_2]$ will never be able to escape this r-interval, since $r(s)$ will have to change continuously with s and, by the above relation,

$$E^2 - F_{L,\mu}(r(s)) = \dot{r}^2(s) \geqslant 0 \quad \text{for all } s.$$

We can rearrange the terms in the expression $h(r) = E^2 - F_{L,\mu}(r)$ as follows:

$$h(r) = (E^2 - \mu) + \frac{1}{r}\Big(2M\mu - \frac{L^2}{r}\Big) + \frac{2M}{r^3}L^2.$$

From the above expression, it can be easily seen that, for any $R > 0$ sufficiently large and $0 < \epsilon < \frac{4M\mu}{3R}$ sufficiently small, if $E^2 = \mu - \epsilon$ and L^2 is chosen to satisfy

$$\frac{M\mu - \frac{1}{4}R\epsilon}{1 - \frac{4M}{R}} < \frac{L^2}{R} < \frac{2M\mu - R\epsilon}{1 - \frac{2M}{R}}$$

(note that the above range is indeed non-empty if R is sufficiently large and ϵR sufficiently small) then the function h has the property that there exist $r_1 \in (\frac{R}{2}, R)$ and $r_2 > R$ such that

$$h(r) \geqslant 0 \text{ on } [r_1, r_2], \quad h(r) < 0 \text{ on } [\frac{R}{2}, r_1) \cup (r_2, +\infty).$$